Novel Estimation of the Thermal Conductivity of Superconducting Niobium

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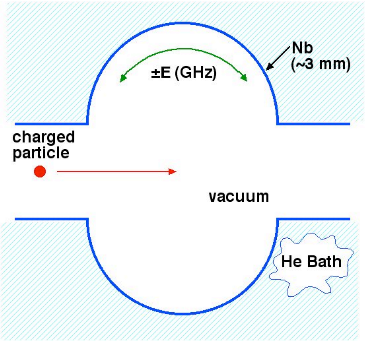
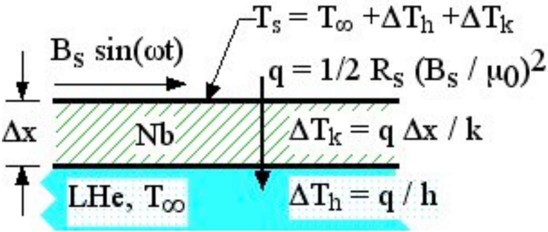
Basic Function of Accelerator



- Superconducting Radio Frequency (SRF) cavities accelerate charged particles to 5 to 80% of c using an RF electric field
- Niobium is used to construct SRF cavities due to its high T_c (9.25 K)
- SRF cavities usually operate at 2 to 4.2 K

Power dissipation

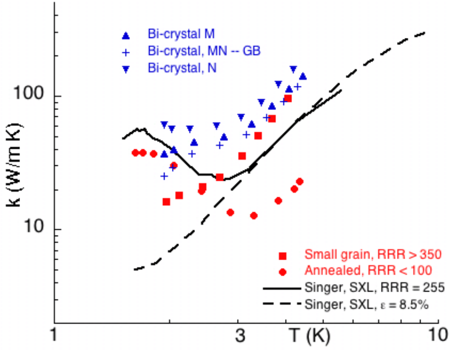
- The RF field induces magnetic field inside the cavity
- The magnetic field heats the inner surface of the cavity wall
- The heat needs to be dissipated into surrounding liquid helium to maintain the bulk temperature below T_c
- Imperfections in the Nb surface result in local heating



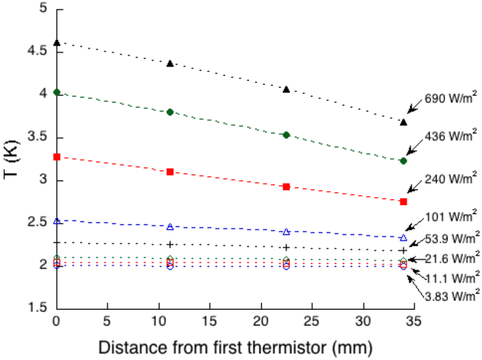
Thermal Conductivity of Superconducting Nb

Conduction in Nb at these temperatures is a function of

- purity
- imperfection density
- grain orientation
- grain boundaries



Temperature data



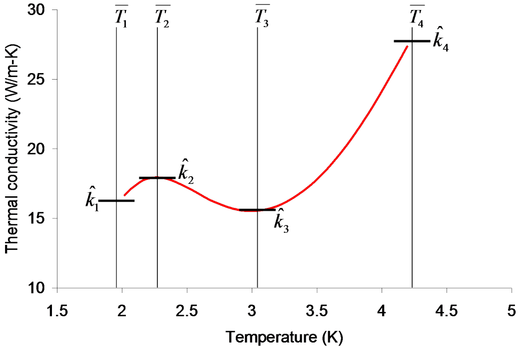
- Points represent thermistor temperature measurements.
- Dotted lines assume uniform conductivity between the sensors.

Each experiment yields a thermal conductivity estimated by

$$k = -\frac{q''}{dT/dx}$$

Thermal Conductivity Estimation

To make simultaneous use of all temperature measurements, an alternate model to estimate conductivity is proposed.



- A set of m temperatures $\bar{T}_p$ are specified
- Thermal conductivity $\hat{k}_p$ is estimated at $\bar{T}_p$

Fourier Law is,

$$q'' = -k_{avg}\frac{dT}{dx} \tag{1}$$

Assuming a linear variation of k_{avg} with respect to T,

$$k_{avg} = \hat{k}_p + \left(\frac{\left(\frac{T_{i,j}+T_{i,j+1}}{2}\right) - \bar{T}_p}{\bar{T}_{p+1} - \bar{T}_p}\right)(\hat{k}_{p+1} - \hat{k}_p) \tag{2}$$

where $T_{i,j}$ is the measured temperature for i^{th} experiment and j^{th} sensor location, and is assumed to be known and error-less. $T_{i,j+1}$ is the temperature at sensor location $j+1$, which is to be estimated.

Hence, Eq. 1 can be re-written as

$$q_i'' = \frac{T_{i,j} - T_{i,j+1}}{dx}\left(\hat{k}_p + \frac{\left(\frac{T_{i,j}+T_{i,j+1}}{2}\right) - \bar{T}_p}{\bar{T}_{p+1} - \bar{T}_p}(\hat{k}_{p+1} - \hat{k}_p)\right) \tag{3}$$

Re-arranging Eq. 3, gives a quadratic in $T_{i,j+1}$

$$A(T_{i,j+1})^2 + B(T_{i,j+1}) + C = 0 \tag{4}$$

where,

$$A = \frac{\hat{k}_{p+1} - \hat{k}_p}{2(\bar{T}_{p+1} - \bar{T}_p)} \qquad B = \hat{k}_p - \frac{\bar{T}_p(\hat{k}_{p+1} - \hat{k}_p)}{\bar{T}_{p+1} - \bar{T}_p}$$

$$C = q_i'' dx - T_{i,j}\hat{k}_p - \frac{T_{i,j}(T_{i,j} - 2\bar{T}_i)(\hat{k}_{p+1} - \hat{k}_p)}{2(\bar{T}_{p+1} - \bar{T}_p)}$$

The solution for Eq. 4 is

$$T_{i,j+1} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$T_{i,j}$ may also be written as a Taylor series expansion, i.e.

$$
\begin{aligned}
T_{i,j}^{(z+1)} = \; & T_{i,j}^{(z)}\Big|_{\hat{k}_1^{(z)},\ \hat{k}_2^{(z)},\ldots,\ \hat{k}_m^{(z)}} \\
& + \frac{\partial T_{i,j}^{(z)}}{\partial \hat{k}_1}\Bigg|_{\hat{k}_2^{(z)},\ \hat{k}_3^{(z)},\ldots,\ \hat{k}_m^{(z)}} \left(\hat{k}_1^{(z+1)} - \hat{k}_1^{(z)}\right) \\
& + \frac{\partial T_{i,j}^{(z)}}{\partial \hat{k}_2}\Bigg|_{\hat{k}_1^{(z)},\ \hat{k}_3^{(z)},\ldots,\ \hat{k}_m^{(z)}} \left(\hat{k}_2^{(z+1)} - \hat{k}_2^{(z)}\right) \\
& + \ldots + \frac{\partial T_{i,j}^{(z)}}{\partial \hat{k}_m}\Bigg|_{\hat{k}_1^{(z)},\ \hat{k}_2^{(z)},\ldots,\ \hat{k}_{m-1}^{(z)}} \left(\hat{k}_m^{(z+1)} - \hat{k}_m^{(z)}\right)
\end{aligned}
\tag{5}
$$

The sensitivity coefficients are determined numerically by

$$\frac{\partial T_{i,j}(x, \hat{k}_1^{(z)}, \ldots, \hat{k}_m^{(z)})}{\partial \hat{k}_m} \approx \frac{T_{i,j}(x, \hat{k}_1^{(z)}, \ldots, d * \hat{k}_m^{(z)}) - T_{i,j}(x, \hat{k}_1^{(z)}, \ldots, \hat{k}_m^{(z)})}{d * \hat{k}_m^{(z)} - \hat{k}_m^{(z)}} \tag{6}$$

where $d = 1.0001$, as an example.

The sum of squares with Tikhonov regularization is

$$S_T = \sum_{i=1}^{r}\sum_{j=1}^{\ell}(Y_{i,j} - T_{i,j}^{(z+1)})^2 + \alpha_T \sum_{p=1}^{m}(\hat{k}_{p+1}^{(z+1)} - \hat{k}_p^{(z+1)})^2 \qquad (7)$$

Partially differentiating with respect to each k_p and setting equal to zero,

$$\frac{\partial S_T}{\partial \hat{k}_p} = -2\sum_{i=1}^{r}\sum_{j=1}^{\ell}(Y_{i,j} - T_{i,j}^{(z+1)})\left(\frac{\partial T_{i,j}^{(z+1)}}{\partial k_p}\right) - 2\alpha_T \Delta\hat{k}_p^{(z+1)} = 0 \qquad (8)$$

where

$$\Delta\hat{k}_p^{(z+1)} = \begin{cases} \hat{k}_p^{(z+1)} - \hat{k}_{p+1}^{(z+1)} & \text{if } p = 1 \\ \hat{k}_{p-1}^{(z+1)} - 2\hat{k}_p^{(z+1)} + \hat{k}_{p+1}^{(z+1)} & \text{if } 1 < p < m \\ \hat{k}_{p-1}^{(z+1)} - \hat{k}_p^{(z+1)} & \text{if } p = m \end{cases}$$

Linearizing Eq. 8 in terms of the corrections in the $(z+1)^{st}$ iteration gives

$$\sum_{i=1}^{r}\sum_{j=1}^{\ell}\left(Y_{i,j} - T_{i,j}^{(z)} - \sum_{s=1}^{m+1}\frac{\partial T_{i,j}^{(z)}}{\partial \hat{k}_s}\Delta\hat{k}_s^{(z)}\right)\frac{\partial T_{i,j}^{(z)}}{\partial \hat{k}_p} + 2\alpha_T\Delta\hat{k}_p^{(z+1)} = 0 \quad (9)$$

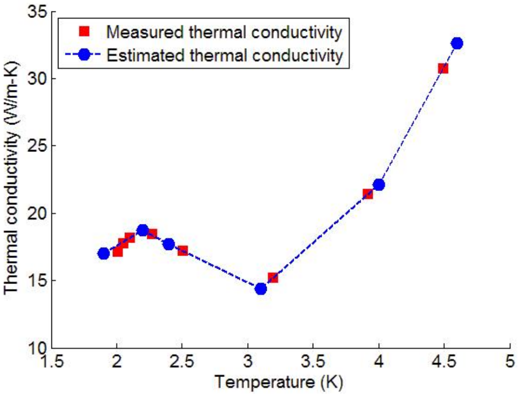
where

$$\Delta\hat{k}_s^{(z)} = \hat{k}_s^{(z+1)} - \hat{k}_s^{(z)}$$

and $\Delta\hat{k}_p^{(z+1)}$ as defined before.

A set of simultaneous algebraic equations for $p = 1, 2, \ldots, m$ is constructed from Eq. 9 and solved for the $(z+1)^{st}$ iteration for the thermal conductivity parameters.

Comparison of Estimates



- $\hat{k}_p$ estimates compare well with k
- Tikhonov regularization parameter $\alpha = 10^{-8}$ for this estimation

Discussion

- New method provides a means for describing complex behavior
- Agrees well with point-wise estimation of k
- Readily expandable to include more $\bar{T}_p$ and $\hat{k}_p$
- Tikhonov parameter is small, as expected for small number of parameters
- Could be used for other phenomena that have no known physical model

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